length of the vehicle and $\phi_o = 0$ or π according to whether the vehicle's reference flight speed, U_o, is less than or greater than orbital speed at the reference flight altitude, h. Furthermore, $\hat{t} = t/t^*$, where t is the time and $t^* = L/2U_o$. To retain conciseness here, the reader is requested to refer to Ref. 1 for further definitions of the symbols used in Eqs. (1) and (2).

Note that the dynamic Eqs. (1) for χ, β, $\hat{p}$, and $\hat{r}$ are uncoupled from the kinematic Eqs. (2) for the lateral center of mass displacement and yaw perturbation angle, as are the equations for ϕ, β, $\hat{p}$, and $\hat{r}$ in the conventional flat-Earth equations.[2] Of course, Eqs. (2) are slightly more complex than their flat-Earth counterparts.

An obvious solution to Eqs. (1) and (2) is

$$\chi = \beta = \hat{p} = \hat{r} = 0 \tag{3}$$

$$\bar{y}_1 = \bar{y}_1{}^* = a_o \cos \hat{\omega}\hat{t} + b_o \sin \hat{\omega}\hat{t} \tag{4a}$$

$$\psi = \psi^* = -a_o \sin \hat{\omega}\hat{t} + b_o \cos \hat{\omega}\hat{t} \tag{4b}$$

where a_o and b_o are constants of integration and $\hat{\omega} = L/(2R_e)$. Since $\phi = \bar{y}_1 + \chi$, in this special case, $\phi = \bar{y}_1 = \bar{y}_1{}^*$. This is the new oscillatory mode cited in Ref. 1.

The exact period of the reference great-circle motion is $2\pi/\Omega$, where $\Omega = U_o/(R_e + h)$, but Drummond used the approximation $\Omega \simeq U_o/R_e$. The period of the oscillatory motion defined by Eqs. (4) is also $2\pi R_e/U_o$.

By referring to Fig. 2 and recalling that $\bar{y}_1 = y_1/R_e$, it is easily seen that the motion corresponding to Eqs. (4) and $\phi = \bar{y}_1$ is *steady flight* in a great circle which is rotated with respect to the original one through an angle, $I = (a_o{}^2 + b_o{}^2)^{1/2}$. The maximum value of $\bar{y}_1{}^*$ (and ϕ) occurs at point A of Fig. 2 and the maximum ψ^* occurs at point B.

It should be noted that if the approximations $U_o/(R_e + h) \simeq U_o/R_e$, $g = g_o/\{1 + [\bar{y}_1/(1+h)]^2\} \simeq g_o$, and $\varepsilon = y_1/(R_e + h) \simeq y_1/R_e$ had not been used in Ref. 1, then if a "free-satellite" condition happened to exist, the period of the oscillatory mode would be exactly equal to the orbital period.

Suppose that perturbations in the variables, χ, β, $\hat{p}$, and $\hat{r}$, occur at $\hat{t} = 0$. Suppose also that the solution to Eqs. (1) is asymptotically stable. It then follows, by using Eqs. (2), that the resulting steady-state motion of the vehicle will, in general, correspond to motion on a "perturbed" great circle, which is rotated with respect to the reference great-circle, and the vehicle ends up in the undamped oscillatory mode. The displacement of the steady-state trajectory in this case is quite analogous to that predicted by flat-Earth lateral stability analyses.[3]

The equations given here may be easily reduced to their flat-Earth counterparts. If the flat-Earth approximation, $R_e \to \infty$, is made and we set $\phi_o = 0$, then $\chi = \phi$, $d\psi/d\hat{t} = (L/b)\hat{r}$, and $d\hat{y}/d\hat{t} = \beta + \psi$, where $\hat{y} = 2y_1/L$. Furthermore, the rest of the usual flat-Earth equations (Ref. 2 with $\gamma_e = 0$) follow from Eqs. (1).

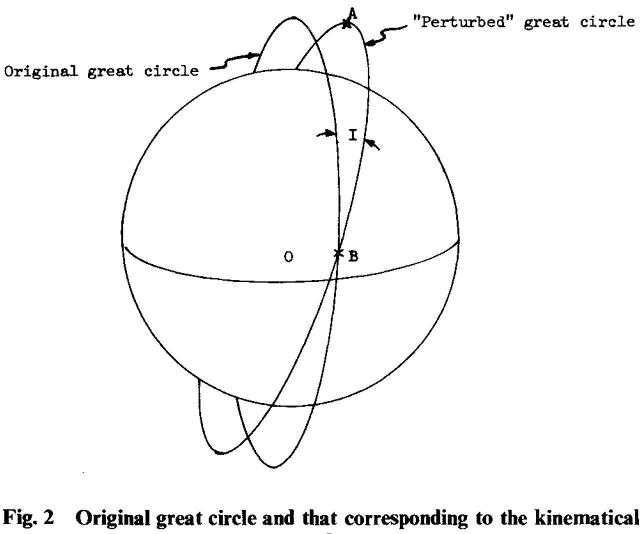

Fig. 2 Original great circle and that corresponding to the kinematical mode.

Conclusion

Linear perturbation equations for the lateral dynamics of flight on a great circle have been presented. The form of these equations is such that the vehicle dynamics are uncoupled from the flight path kinematics in a manner analogous to the corresponding flat-Earth equations. It has been shown that the new oscillatory mode cited in Ref. 1 corresponds to steady flight on a great circle which is, in general, not the reference flight path. In fact, for a dynamically stable vehicle, the steady-state motion following a perturbation will be motion in a "perturbed" great circle with the period of the new oscillatory mode.

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Finite-Difference Version of Quasi-Linearization Applied to Boundary-Layer Equations

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Introduction

The analysis of the boundary-layer equation is characterized by the split two-point boundary conditions. The feature presents an essential difficulty in numerical calculation of the boundary-layer problem. This difficulty appears both in similar flows, which are governed by one or more ordinary differential equations and in nonsimilar flows, which are governed by partial differential equations.

There have been a number of methods to overcome this difficulty. Among them, the quasi-linearization method developed by Radbill[1] seems to give a very powerful tool for treating the split boundary conditions. The method converts the nonlinear two-point boundary value problem into an iterative scheme of solution which involves the step-by-step integration of linear differential equations with two-point boundary conditions. The conditions at both boundaries are conserved and satisfied at every iteration.

In some circumstances, the finite-difference scheme is considered more desirable than the scheme of type of step-by-step integration. Lew presented an application of finite-difference scheme to boundary-layer equation.[2] His method consists of the accelerated replacement solution of simultaneous nonlinear algebraic equations. But the number of iteration cycles necessary for the prescribed criteria of the accuracy is rather large. Although using the acceleration parameter ω, in case of the two-dimensional stagnation point flow, for example, 47 iterations were necessary to obtain the solution within the given accuracy criterion $\varepsilon = 10^{-4}$. This seems the fatal point of the method.

In the present Note, we describe a finite-difference version of the quasilinearization method applied to boundary-layer equation. The method eliminates the estimation of the accelera-

Received August 30, 1973.
Index category: Boundary Layers and Convective Heat Transfer—Laminar.

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tion parameter ω in Lew's method. And the result for the same example as his shows a rapid convergence to the desired solution from a crude initial guess.

Method of Solution

We proceed the discussion to outline the method along with the application to the solution of the Falkner-Skan equation. Then, consider the well-known equation

$$f''' + ff'' + \beta(1 - f'^2) = 0 \tag{1}$$

satisfying the split boundary conditions

$$f(0) = f'(0) = 0 \qquad f'(\infty) = 1$$

where the independent variable is the usual similarity variable η. Equation (1) is equivalent to a system of three first-order equations, namely

$$f' = \xi \qquad \xi' = G \qquad G' = -fG - \beta(1-\xi^2) \tag{2}$$

subject to $f(0) = \xi(0) = 0$ and $\xi(\infty) = 1$. Applying the idea of quasi-linearization[1,3] to Eqs. (2), we obtain a set of linear equations

$$f'^{(k+1)} = \xi^{(k+1)} \tag{3}$$

$$\xi'^{(k+1)} = G^{(k+1)} \tag{4}$$

$$G'^{(k+1)} = -f^{(k)}G^{(k+1)} + 2\beta\xi^{(k)}\xi^{(k+1)} - \beta[1 + (\xi^{(k)})^2] - G^{(k)}(f^{(k+1)} - f^{(k)}) \tag{5}$$

where k and $k+1$ are the iteration indices.

When the iterative approximation proceeds, the last term on the right-hand side of Eq. (5) tends to zero. Of course the neglect of it influences the rate of convergence of the iteration but does not need any essential change in the idea of the method of approximation. Dropping this from the right-hand side of Eq. (5) and combining with Eq. (4), we get a second-order equation for the function $\xi^{(k+1)}$

$$\xi''^{(k+1)} + f^{(k)}\xi'^{(k+1)} - 2\beta\xi^{(k)}\xi^{(k+1)} = -\beta[1 + (\xi^{(k)})^2] \tag{6}$$

where it is assumed that the functions $f^{(k)}$ and $\xi^{(k)}$ are known. This formulation is necessary to eliminate the need for third-order difference and to obtain a tridiagonal system of linear algebraic equations at further stage of the analysis. The boundary conditions for Eq. (6) are

$$\xi^{(k+1)}(0) = 0 \tag{7}$$

and

$$\xi^{(k+1)}(\infty) = 1$$

In the numerical analysis, we replace the latter by

$$\xi^{(k+1)}(\eta_e) = 1 \tag{8}$$

where η_e is a sufficiently large value of η.

Now, we write the problem in a finite-difference form. We divide the interval $0 \leqq \eta \leqq \eta_e$ into $(N-1)$ equal intervals and denote the values of the dependent variables at $\eta_i = (i-1)h$ with the subscript $i\,(=1, 2, \ldots, N)$, where $h = \eta_e/(N-1)$. Substituting, as usual, the expressions

$$\xi_i'' = (\xi_{i+1} - 2\xi_i + \xi_{i-1})/h^2 \qquad \xi_i' = (\xi_{i+1} - \xi_{i-1})/2h$$

into Eq. (6) and using the boundary conditions (7) and (8), we obtain a system of linear algebraic equations in a tridiagonal form

$$A_{22}\xi_2^{(k+1)} + A_{23}\xi_3^{(k+1)} = C_2 \tag{9a}$$

$$A_{i,i-1}\xi_{i-1}^{(k+1)} + A_{i,i}\xi_i^{(k+1)} + A_{i,i+1}\xi_{i+1}^{(k+1)} = C_i \qquad (i = 3, 4, \ldots, N-2) \tag{9b}$$

$$A_{N-1,N-2}\xi_{N-2}^{(k+1)} + A_{N-1,N-1}\xi_{N-1}^{(k+1)} = C_{N-1} - A_{N-1,N} \tag{9c}$$

where

$$A_{i,i-1} = 2 - hf_i^{(k)} \qquad A_{i,i} = -4(1 + h^2\beta\xi_i^{(k)})$$

$$A_{i,i+1} = 2 + hf_i^{(k)} \qquad C_i = -2h^2\beta[1 + (\xi_i^{(k)})^2]$$

The system (9) is composed of $(N-2)$ equations for $(N-2)$ unknowns $\xi_i^{(k+1)}$. It can be solved quite easily by usual sweeping method. Once all of $\xi_i^{(k+1)}$ are determined, $f_i^{(k+1)}$ are obtained from Eq. (3), namely

$$f_i^{(k+1)} = \int_0^{\eta_i} \xi_i^{(k+1)} d\eta$$

executing a numerical integration. The values of $\xi_i^{(k+1)}$ and $f_i^{(k+1)}$ obtained here are used to replace $\xi_i^{(k)}$ and $f_i^{(k)}$ for the next cycle. The convergence is considered achieved if $|\xi_i^{(k+1)} - \xi_i^{(k)}| < \varepsilon$ for all points, where ε is a prescribed accuracy criterion.

Numerical Example

A sample calculation was done for the case of $\beta = 1$. This was chosen to permit direct comparison with the results of Refs. 1 and 2. The calculation was carried out with $\eta_e = 5$, $h = 0.05$, and $\varepsilon = 10^{-4}$. We assumed for the first iterate the same discontinuous solution as Ref. 1; $f_i^{(0)} = (1/8)\eta_i^2$ and $\xi_i^{(0)} = (1/4)\eta_i$ for $0 \leqq \eta \leqq 4$, and $f_i^{(0)} = \eta_i - 2$ and $\xi_i^{(0)} = 1$ for $\eta > 4$.

Profiles of $\xi = f'$, $G = \xi' = f''$ and f for every stage of the iteration are shown on Figs. 1 and 2. The values of $f''(0)$ obtained at each iteration are tabulated in Table 1. In spite of the crudity of the initial solution, we observe there a rapid convergence of the iteration to the final solution. On these figures, the curves for the second, third and fourth iterates are indistinguishable from those for the final (fifth) one. The values of all three functions in the final approximation differed from

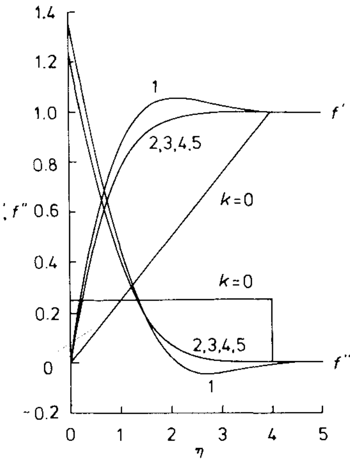

Fig. 1 Convergence of approximations for velocity and shear function.

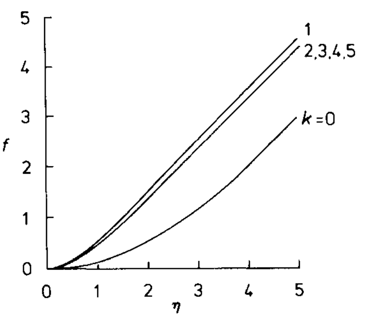

Fig. 2 Convergence of approximations for stream function.

Table 1 Convergence of approximations for the shear stress $f''(0)$

k	$f''(0)$
0	0.25
1	1.3354616
2	1.2427286
3	1.2333815
4	1.2327186
5	1.2326740

the corresponding published values in Ref. 4 by less than 5 in the fourth decimal place at each of the 36 tabulated points with the calculated value $f''(0)^{(5)} = 1.2326740$ as compared with 1.232588.

The sample calculations other than for the case of $\beta = 1$ were carried out for the cases of $\beta = 0.5$ and 0 using the same initial approximations and the same values of η_e, h and ε as for the case of $\beta = 1$. The converged solutions were obtained after 4 and 8 iterations, respectively.

Unfortunately, the present method cannot be applied to a problem in which the uniqueness of the solution is not assured. For such a problem, special account of the asymptotic behavior of the solution at the outer edge of the boundary layer must be taken and an improved version of quasi-linearization by Libby and Chen[3] seems to be the best of the existing methods. The present method, despite the restriction, seems to provide another useful technique for treating the two-point boundary value problem and is also applicable to the numerical analysis of nonsimilar flows by a difference-differential scheme, such as has been developed by Smith and Clutter.[5]

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Variations of Eigenvalues and Eigenfunctions in Continuum Mechanics

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I. Introduction

THE main procedure of a typical design problem usually consists of repeated modifications of design parameters and the investigation of the system response for each set of these parameters to achieve certain system performance, corresponding to favorite design. A more ambitious aim may be to achieve the best design which optimizes the response of the system. Whatever the goal of design may be, the repeated analysis is, in nature, costly and, hence, there exists an increasing need for the prediction of the system response, due to changes in design parameters, in a more efficient way. The literature contains some attempts to the end of prediction of the response of a system with new set of design parameters from the response of the old system. (See for example Refs. 1–6.) Attempts of this sort have been, so far, directed towards the study of discrete models according to which the system can be designed and synthesized by a set of numbers, the so called "design parameters." A more realistic and refined idea of engineering design would be the characterization of systems by a set of "design functions." Such a consideration finds a vital importance in the true optimization problems in which the perfect design can only be achieved through the investigation of systems with distributed properties. With the above notions in mind, the present work is aimed at the prediction of the change in response of continuous systems caused by the variations of design functions. We shall obtain a set of relations which give the firsit-order variations of the eigenvalues and eigenfunctions of the new design in terms of these quantities belonging to the previous design. As illustrative examples, several classes of problems in vibration, stability and optimization are treated.

Received September 17, 1973.
Index categories Aircraft Vibration; Structural Design, Optimal; Structural Stability Analysis.

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II. Variation of Eigenvalues

Consider the following continuous system of eigenvalue problem:

$$\mathbf{LY} + \lambda\mathbf{MY} = 0 \tag{1}$$

where $\mathbf{Y}$ is the state vector and $\mathbf{L}$ and $\mathbf{M}$ are, in matrix form, two linear differential operators and embody the functions $h_i(x)$ characterizing the system, that is

$$\mathbf{L} = \mathbf{L}[h_i(x)], \qquad \mathbf{M} = \mathbf{M}[h_i(x)] \tag{2}$$

these field equations are to be accompanied by a set of appropriate boundary conditions. The above system is allowed to be nonself adjoint in the general case. Suppose that λ_m and $\mathbf{Y}_m$ $(m = 1, 2, \ldots)$ are the eigenvalues and the corresponding eigenfunctions of the system and let $\mathbf{Y}_n{}^*$ be the eigenfunction of the associated adjoint system. The biorthogonality condition of Eq. (1) and its adjoint can be expressed as[7]

$$\int_D \mathbf{Y}_n{}^{*\prime}\mathbf{MY}_m dD = (\mathbf{Y}_n{}^*, \mathbf{MY}_m) = \delta_{mn} = \begin{vmatrix} 0 & m \perp n \\ 1 & m = n \end{vmatrix} \tag{3}$$

wherein D is the domain of integration and $\mathbf{Y}_n{}^{*\prime}$ is the transpose of $\mathbf{Y}_n{}^*$.

If the functions $h_i(x)$ characterizing the system are varied, the characteristic response of the system such as its eigenvalues and eigenfunctions will also change. To the end of determination of the variation of these quantities, denoted by $\delta\lambda_m$ and $\delta\mathbf{Y}_m$, due to variation of $h_i(x)$, denoted by $\delta h_i(x)$, we do as follows:

The Rayleigh quotient corresponding to the eigenfunction $\mathbf{Y}_m$ of the system[1] is

$$(\mathbf{Y}_m{}^*, \mathbf{LY}_m) + \lambda_m(\mathbf{Y}_m{}^*, \mathbf{MY}_m) = 0 \tag{4}$$

If we take the first variation of both sides of Eq. (4), with the help of relations (1) and (3), we get

$$\delta\lambda_m = -[(\mathbf{Y}_m{}^*, \delta\mathbf{LY}_m) + \lambda_m(\mathbf{Y}_m{}^*, \delta\mathbf{MY}_m)] \tag{5}$$

Note that the variation symbol, δ, in front of a letter implies the variation of that letter only and not the subsequent letters. Equation (5) yields the variation of the eigenvalue λ_m, due to variation of $h_i(x)$, in terms of already known eigenvalues and eigenfunctions of the original problem with unperturbed $h_i(x)$.

III. Variation of Eigenfunctions

If we take the first variation of both sides of Eq. (1), we obtain

$$\delta\mathbf{LY}_m + L\,\delta\mathbf{Y}_m + \delta\lambda_m\mathbf{MY}_m + \lambda_m\,\delta\mathbf{MY}_m + \lambda_m\mathbf{M}\,\delta\mathbf{Y}_m = 0 \tag{6}$$

Now, considering the fact that the eigenfunctions $\{\mathbf{Y}_m\}$ form a complete set we can expand the variation of $\mathbf{Y}_m$ in terms of these eigenfunctions, so we have

$$\delta\mathbf{Y}_m = \sum_j a_{mj}\mathbf{Y}_j \tag{7}$$

Substituting the above expansion into Eq. (6) and taking the inner product, according to Eq. (3), of both sides of the resulting equation with $\mathbf{Y}_n{}^*$, we obtain

$$(\mathbf{Y}_n{}^*, \delta\mathbf{LY}_m) + (\mathbf{Y}_n{}^*, \sum_j a_{mj}\mathbf{LY}_j) + \delta\lambda_m(\mathbf{Y}_n{}^*, \mathbf{MY}_m) + \lambda_m(\mathbf{Y}_n{}^*, \delta\mathbf{MY}_m) + \lambda_m(\mathbf{Y}_n{}^*, \sum_j a_{mj}\mathbf{MY}_j) = 0 \tag{8}$$

Finally the substitution of j for n and the utilization of Eqs. (1) and (3) yields

$$a_{mj} = [-1/(\lambda_m - \lambda_j)][(Y_j{}^*, \delta LY_m) + \lambda_m(Y_j{}^*, \delta MY_m)] \quad m \neq j \tag{9}$$

to get a_{mm} we use Eq. (3), for the case of self-adjoint systems and we find